PERGAMON

International Journal of Solids and Structures 40 (2003) 4033–4041

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Stress trajectories in the presence of friction

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Received 5 June 2002; received in revised form 10 March 2003

Abstract

We study the stress field around an oblique closed defect in polymethylmethacrylate (PMMA) plates under uniaxial compression in the presence of friction. Every specimen (PMMA plate) with a slot may be compared to an infinite plate with an elliptical cavity where the long axis is equal to the slot length. We have calculated the stress tensor for an elliptical oblique closed slot using the system of curvilinear co-ordinates defined by two families of ellipses and homofocal hyperbolas and we have plotted the isovalues and the stress trajectories in terms of the friction intensity.

Keywords: Stress; Fracture; Defect; PMMA; Compression; Friction

1. Introduction

An increasing number of geological observations have shown that some kinds of fault on different scales originate from shearing movements along pre-existing joints. Branch cracks have been observed in various materials such as polymers (Bowden, 1970; Poirier, 1980; Brun and Cobbold, 1980), plaster (Lajtai, 1971) and glass (Bombolakis, 1964, 1968). The branch crack also occurs in rocks such as granite (Ashby and Sammis, 1988) and at the tip of geological faults.

The aim of this paper is to study the stress field linked to the defect, in a polymethylmethacrylate (PMMA) plate with a slot, whose axis makes an angle β ranging from 15° to 75° with the direction of imposed uniaxial compressive stress. Very thin plates of different materials were inserted within the slot to ensure a known friction coefficient and in order to have an oblique closed slot (Chaker and Barquins, 1996; Barquins et al., 1997). Under uniaxial compression, the relative displacement of initial surface defect corresponds to mode II.

2. Mathematical problem

To study the mathematical models, the PMMA plate may be compared to an infinite plate with an elliptical cavity where the long axis is equal to the slot length (Inglis, 1913; Westergaard, 1939; Irwin, 1957;

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0020-7683/03/$ - see front matter
doi:10.1016/S0020-7683(03)00173-2

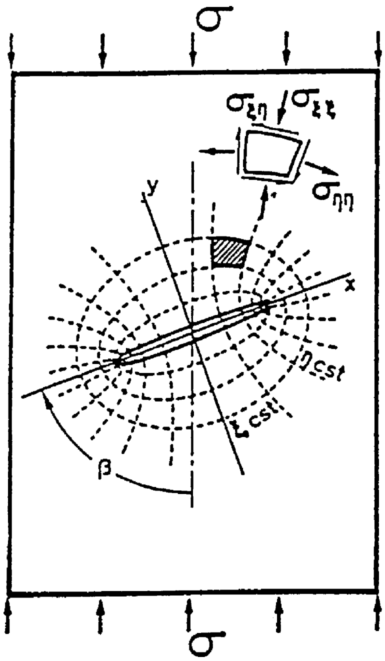

Fig. 1. State of stress near an elliptical crack tip in curvilinear co-ordinates.

Sneddon, 1964). The stress tensor for an elliptical oblique closed slot can be calculated using the system of curvilinear co-ordinates defined by two families of ellipses and homofocal hyperbolas (Fig. 1). For an open slot, the stress tensor was found by Wu and Chang (1978). A very powerful method for solving plane problems was developed first by Stevenson (1945), over by Muskhelishvili (1953), after by Timoshenko and Goodier (1961) and Jeager and Cook (1979).

3. Stress tensor

In order to calculate the stress tensor for an elliptical oblique closed slot, we use the method of complex analysis:

$$\Phi = \mathrm{Re}[\bar{z}\psi(z) + \chi(z)] \tag{1}$$

involves finding two analytic functions $\psi(z)$ and $\chi(z)$ of the complex variable $z = x + \mathrm{i}y$. These functions are related to the stresses and displacements by these equations:

- stresses:

$$\begin{cases} \sigma_x + \sigma_y = 2[\psi'(z) + \bar{\psi}'(\bar{z})] = 4\,\mathrm{Re}\,\psi'(z) \\ \sigma_y - \sigma_x + 2\mathrm{i}\sigma_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)] \end{cases} \tag{2}$$

- displacement:

$$u_x + \mathrm{i}u_y = (1/2\mu)[k\psi(z) - z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})] \tag{3}$$

where $k = 3 - 4v$ in plane deformation

$$k = \frac{3-v}{1-v} \quad \text{in plane stress}$$

and v is the Poisson's ratio. After, we take into account the boundary conditions, then we give the stress vector (t_x, t_y) on the exterior outline $\partial\Omega$, the functions $\psi(z)$ and $\chi(z)$ must satisfy the limit condition where $z \in \partial\Omega$ and s is the curvilinear abscise:

$$t_x + \mathrm{i}t_y = -\mathrm{id}[\psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}(\bar{z})]/\mathrm{d}s \tag{4}$$

The resolution of the general elasticity plane problem consist to find two analytic functions $\psi(z)$ and $\chi(z)$ satisfy the limit condition in order to determine the stress tensor around an elliptical slot. The limit condition (4) can be changed, when z tend to the infinite, with giving this asymptotic development:

$$\begin{cases} \psi'(z) \approx \Gamma_0 + \mathrm{O}(1/z^2) \\ \chi'(z) \approx \Gamma + \mathrm{O}(1/z^2) \end{cases} \qquad |z| \to +\infty \tag{5}$$

where Γ_0 and Γ are the stresses connected to the infinite uniform stresses σ_x^∞, σ_y^∞ and σ_{xy}^∞. Then, with Eq. (2) we obtain:

$$\begin{cases} \Gamma_0 = (\sigma_1^\infty + \sigma_2^\infty)/4 = (\sigma_x^\infty + \sigma_y^\infty)/4 \\ \Gamma = (\sigma_2^\infty - \sigma_1^\infty)\mathrm{e}^{-2\mathrm{i}\beta}/2 = (\sigma_y^\infty - \sigma_x^\infty)/2 + \mathrm{i}\sigma_{xy} \end{cases} \tag{6}$$

where σ_1^∞ and σ_2^∞ are the principles stresses at infinite and β is the first principle direction angle at infinite with the abscises axe (Ox). Finally, in presence of defect ξ supposed free $(t_x + \mathrm{i}t_y = 0)$ the condition (4) becomes:

$$\psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}(\bar{z}) = \text{constant} \quad \text{where } z \in \xi \tag{7}$$

The two analytic functions $\psi(z)$ and $\chi(z)$ must satisfy the two conditions (5) and (7). Then, we apply the principle of superposition, and we treat the case of infinite plane with a closed defect where the major axis makes an angle β with the direction of uniaxial load. In the first time, we suppose that the orientation of the cartesian system where the origin is the centre of the ellipse, such us the axis (Ox) and (Oy) coincide with a and b of the ellipse (Fig. 1). After, we introduce the elliptical co-ordinates (ξ, η) with these transformation equations:

$$\begin{cases} x = c\cosh\xi\cos\eta \\ y = c\sinh\xi\sin\eta \end{cases} \tag{8}$$

where c is equal the focal distance of the elliptic form $(c^2 = a^2 - b^2)$. The two Eq. (8) are equivalent to:

$$z = x + \mathrm{i}y = c\cosh\zeta = c\cosh(\xi + \mathrm{i}\eta) \tag{9}$$

and when we eliminate η in Eq. (8), we get:

$$\frac{x^2}{(c^2\cosh^2\xi)} + \frac{y^2}{(c^2\sinh^2\xi)} = 1 \tag{10}$$

If ξ has the constant value, the corresponding curve in the (x, y) plane is the ellipse, where the half axes are $c\cosh\xi$, $c\sinh\xi$ and the focus are at $x = \pm c$. For different value of ξ, we obtain different ellipses who have the similar focus, i.e. a family of ellipses like presented in Fig. 1. On one of these any ellipses, ξ is constant and η vary on an interval of 2π. In other case, we eliminate ξ between Eq. (8) using the relation: $\cosh^2\xi - \sinh^2\xi = 1$, we get:

$$\frac{x^2}{(c^2\cos^2\eta)} - \frac{y^2}{(c^2\sin^2\eta)} = 1 \tag{11}$$

Similarly, the curves η equal constant are the hyperbolae (Eq. (11)). This equation present a family of homofocal hyperbolae, on one of these any hyperbolae η is constant and ξ vary. The whole of this co-ordinate system is called the elliptic co-ordinate. Every point of the plane (xy) is characterised by the defined value of ξ and η, these value strike the two defined curve by Eqs. (10) and (11) to intercept at this point. If we work in the new curvilinear co-ordinate system, the relation between the stresses and displacements expression at the curvilinear and cartesian co-ordinates systems is given by these equations:

- stresses:

$$\begin{cases} \sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y \\ \sigma_\eta - \sigma_\xi + 2\mathrm{i}\sigma_{\xi\eta} = \mathrm{e}^{2\mathrm{i}\alpha}(\sigma_y - \sigma_x + 2\mathrm{i}\sigma_{xy}) \end{cases} \tag{12}$$

- displacement:

$$u_\xi + \mathrm{i}u_\eta = \mathrm{e}^{-\mathrm{i}\alpha}(u_x + \mathrm{i}u_y) \tag{13}$$

where $\mathrm{e}^{2\mathrm{i}\alpha} = \sinh\zeta / \sinh\bar{\zeta}$.

When we combine Eqs. (2) and (3) with (10)–(13), the stress and displacement components borne by a small element of volume can be expressed by these equation:

- stresses:

$$\begin{cases} \sigma_\xi + \sigma_\eta = 2[\psi'(z) + \bar{\psi}'(\bar{z})] = 4\,\mathrm{Re}[\psi'(z)] \\ \sigma_\eta - \sigma_\xi + 2\mathrm{i}\sigma_{\xi\eta} = 2\mathrm{e}^{2\mathrm{i}\alpha}[\bar{z}\psi'(z) + \chi''(z)] \end{cases} \tag{14}$$

- displacement:

$$u_\xi - \mathrm{i}u_\eta = (1/2\mu)[k\psi(z) - z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})] \tag{15}$$

Far from the notch, the principal stresses are $\sigma_1^\infty = \sigma$ and $\sigma_2^\infty = 0$, then we have with Eqs. (2), (5) and (6):

$$\begin{cases} 4\,\mathrm{Re}[\psi'(z)] = \sigma \\ 2[\bar{z}\psi''(z) + \chi''(z)] = -\sigma\mathrm{e}^{-2\mathrm{i}\beta} \end{cases} \tag{16}$$

and on the defect defined by ($\xi = \xi_0$), the condition (7) become:

$$\psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}(\bar{z}) = \text{constant} \tag{17}$$

where $z \in \xi_0$.

All these limit conditions (16) and (17) can be satisfied when writing the complex functions $\psi(z)$ and $\chi(z)$ in this form:

$$\begin{cases} \psi(z) = (c/4)(A\cosh\zeta + B\sinh\zeta) \\ \chi(z) = (c^2/4)(C\zeta + D\cosh 2\zeta + E\sinh 2\zeta) \end{cases} \tag{18}$$

The complex stress functions in the form put forward by Stevenson (1945) are written by Eq. (18) where A, B, C, D and E are constants whose values are fixed by adequately specified boundary conditions (Eqs. (16) and (17)). The complex functions in the general case (uniaxial and biaxial load) also given by Timoshenko and Goodier (1961) and Jeager and Cook (1979) are written in this equation:

$$\begin{cases} \psi(z) = (\sigma c/4)[n\mathrm{e}^{2(\xi_0+\mathrm{i}\beta)}\cosh\zeta + (m - n\mathrm{e}^{2(\xi_0+\mathrm{i}\beta)}\sinh\zeta)] \\ \chi'(z) = -(\sigma c/4\sinh\zeta)[m\cosh 2\xi_0 - n\cos 2\beta + n\mathrm{e}^{2\xi_0}\sinh 2(\zeta - \xi_0 - \mathrm{i}\beta)] \end{cases} \tag{19}$$

where $m = 1 + k$ and $n = 1 - k$.

For the uniaxial load, $k = 0$ (Chang, 1981a,b), the complex functions $\psi(z)$ and $\chi(z)$ are written:

$$\begin{cases} \psi(z) = (\sigma c/4)[\mathrm{e}^{2\xi_0}\cos 2\beta\cosh\zeta + (1-\mathrm{e}^{2(\xi_0+\mathrm{i}\beta)})\sinh\zeta] \\ \chi(z) = -(\sigma c^2/4)[(\cosh 2\xi_0 - \cos 2\beta)\zeta + (1/2)\mathrm{e}^{2\xi_0}\cosh 2(\zeta-\xi_0-\mathrm{i}\beta)] \end{cases} \tag{20}$$

and we can writing the expressions of all constants A, B, C, D and E with a simple identification between Eqs. (18) and (20):

$$\begin{aligned} A &= \mathrm{e}^{2\xi_0}\cos 2\beta \\ B &= 1-\mathrm{e}^{2(\xi_0+\mathrm{i}\beta)} \\ C &= -(\cosh 2\xi_0 - \cos 2\beta) \\ D &= -(1/2)\mathrm{e}^{2\xi_0}\cosh 2(\xi_0+\mathrm{i}\beta) \\ E &= (1/2)\mathrm{e}^{2\xi_0}\sinh 2(\xi_0+\mathrm{i}\beta) \end{aligned} \tag{21}$$

where $\cosh 2(\zeta-\xi_0-\mathrm{i}\beta) = \cosh 2(\xi_0+\mathrm{i}\beta)\cosh 2\zeta + \sinh 2(\xi_0+\mathrm{i}\beta)\sinh 2\zeta$.

With Griffith theory developed in 1924 (Griffith, 1924), for a defect having an elliptic form and the half axes a and b, the stresses σ_x, σ_y and σ_{xy} are written with principles stresses σ_1 and σ_2 at infinity:

$$\begin{cases} \sigma_x = \sigma_1\sin^2\beta + \sigma_2\cos^2\beta \\ \sigma_y = \sigma_1\cos^2\beta + \sigma_2\sin^2\beta \\ \sigma_{xy} = -1/2(\sigma_1-\sigma_2)\sin 2\beta \end{cases} \tag{22}$$

and the characteristic of the elliptic defect are equal to: $a = c\cosh\xi_0$ and $b = c\sinh\xi_0$. Then, for a closed defect submitted to the uniaxial compression, we are interesting to the stress σ_y given by Eq. (22). When the defect has closed, further displacement can be achieved only by sliding across the closed surface, and this will be resisted by sliding friction across this surface. McClintock and Walsh (1962) first modified the Griffith theory to take this effect into account, and the theory was developed further by Brace (1960), Murrell (1964) and Bombolakis (1973). McClintock and Walsh (1962) assume that a normal stress σ_c at infinity is necessary to close the defect. Then, they assume that a normal stress across the surface of the closed defect (Eq. (23)) and that a frictional force (Eq. (24)) resists sliding across this surface.

$$\sigma_n = \sigma_y - \sigma_c \tag{23}$$

$$\sigma_\rho = \rho\sigma_n \tag{24}$$

then the stress σ_{xy} and the tangential stress σ_t in the crack surface are given respectively by these equations:

$$\sigma_{xy} + \sigma_\rho = \sigma_{xy} + \rho(\sigma_y - \sigma_c) \tag{25}$$

$$\sigma_t = \frac{2\xi_0\sigma_c - 2\eta[\sigma_{xy} + \rho(\sigma_y - \sigma_c)]}{\xi_0^2 + \eta^2} \tag{26}$$

where ρ is a friction coefficient and when we using Eq. (22), Eq. (26) becomes:

$$\sigma_t = \frac{2\xi_0\sigma_c + \eta\sigma^*}{\xi_0^2 + \eta^2} \tag{27}$$

where

$$\sigma^* = (\sigma_1 - \sigma_2)[\sin 2\beta - \rho\cos 2\beta] - \rho(\sigma_1 + \sigma_2 - 2\sigma_c) \tag{28}$$

When we differentiating Eq. (27) with respect to η, σ_t is found to have maximum or minimum values given by this equation:

$$\eta/\xi_0 = \left[-2\sigma_c \pm (4\sigma_c^2 + \sigma^{*2})^{1/2} \right] \Big/ \sigma^* \tag{29}$$

and when we inserting this value of η in Eq. (27), we obtain the extreme values σ_e of σ_t:

$$\sigma_e = \left[2\sigma_c \pm (4\sigma_c^2 + \sigma^{*2})^{1/2} \right] \Big/ 2\xi_0 \tag{30}$$

In Eq. (28), the negative sign is to be taken, since we are interested in tensile values. The maximum value of σ_e as a function of β occurs when $d\sigma^*/d\beta = 0$.

Finally, using Eqs. (14), (15), (23)–(26) and differentiating Eq. (20) with respect to z, we can suggest the expression of the stress tensor for an oblique closed defect with sliding.

4. Stress field

Every specimen (PMMA plate) with a slot, whose axis makes an angle β with the direction of imposed uniaxial compressive stress σ, may be compared to an infinite plate with an elliptical cavity where the long axis is equal to the slot length. Using the system of curvilinear co-ordinates defined by two families of ellipses ξ and homofocal hyperbolas η having the same focus as the ellipse ξ_0 representative of the slot, we can writing the repartition of the stresses in the specimen. The border of the slot is generally smooth, the branch crack has more chance to start at the point where the stress $\sigma_{\eta\eta}(\xi = \xi_0)$ reach his maximum value. The principal stresses σ_1, σ_2 and the plane equivalent stress σ_{eq} are calculated respectively as Eq. (31)–(33):

$$\sigma_1 = (\sigma_\xi + \sigma_\eta)/2 - \{[(\sigma_\xi - \sigma_\eta)/2]^2 + \sigma_{\xi\eta}^2\}^{1/2} \tag{31}$$

$$\sigma_2 = (\sigma_\xi + \sigma_\eta)/2 + \{[(\sigma_\xi - \sigma_\eta)/2]^2 + \sigma_{\xi\eta}^2\}^{1/2} \tag{32}$$

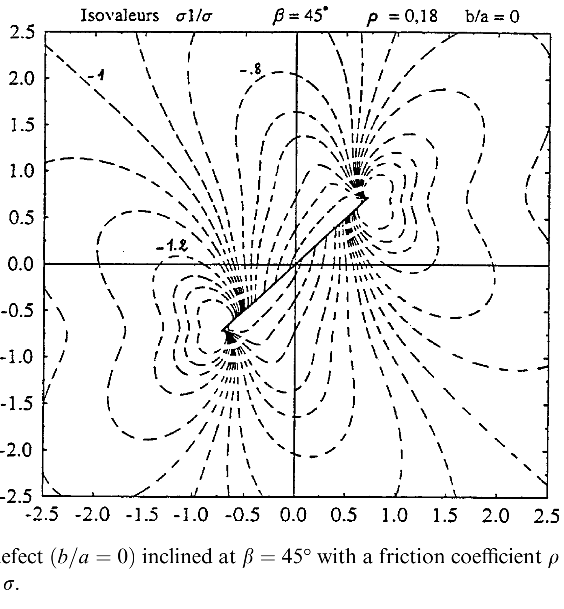

Fig. 2. Stress field around a closed defect ($b/a = 0$) inclined at $\beta = 45°$ with a friction coefficient $\rho = 0.18$. Isovalues of principal stress σ_1 normalised by the applied stress σ.

$$\sigma_{eq} = \{3(\sigma_{\xi\eta})^2 + [(\sigma_{\eta\eta})^2 + (\sigma_{\xi\xi})^2 - \sigma_{\eta\eta}\sigma_{\xi\xi}]/2\}^{1/2} \tag{33}$$

For the defect angle $\beta = 45°$ and the friction coefficient $\rho = 0.18$, we have plotted in the first time the isovalues of the principal stresses σ_1 (Fig. 2), in the second time the isovalues of σ_2 with the corresponding stress trajectories of σ_1 passing through the point of maximum tensile stress σ_2 at the slot edge (Fig. 3), after the isovalues of the plane equivalent stress σ_{eq} (Fig. 4) and finally the stress trajectories of σ_1 and σ_2 (Fig. 5).

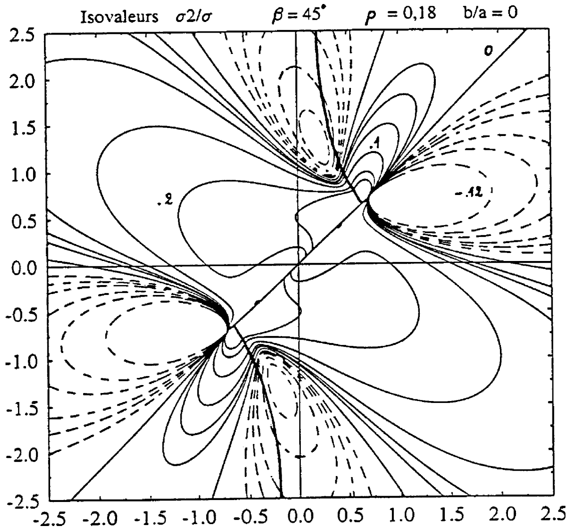

Fig. 3. Isostress values of σ_2/σ for $\beta = 45°$, with superimposition of the stress trajectory of σ_1 touching the point of maximum tensile stress σ_2, for a closed defect ($b/a = 0$) and with a friction coefficient $\rho = 0.18$.

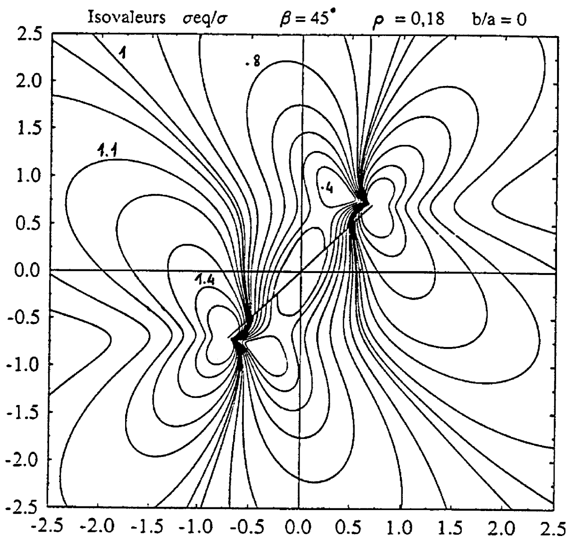

Fig. 4. Stress field around a closed defect ($b/a = 0$) inclined at $\beta = 45°$ with a friction coefficient $\rho = 0.18$. Isovalues of principal stress σ_{eq} normalised by the applied stress σ.

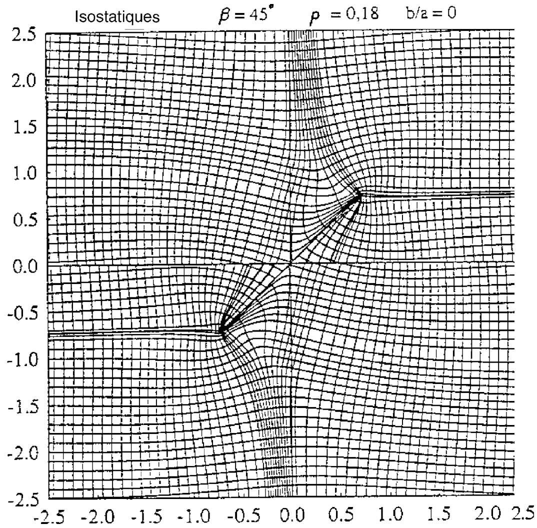

Fig. 5. Stress trajectories of σ_1 and σ_2 for a closed defect ($b/a = 0$) inclined at $\beta = 45°$ with a friction coefficient $\rho = 0.18$.

5. Conclusions

In the first time, we observe that σ_1 is a compressive stress nearly everywhere around the closed defect (dotted lines in Figs. 2 and 3). Whereas σ_2 is a strong tensile stress (heavy lines Fig. 3) in a large area around the slot and is thus the stress which opens the branch crack. Finally, the branch crack will be initiated at the point where the tensile stress σ_2 reaches its maximum value and will propagate along the stress trajectory of the principal stress σ_1 passing at the same point.

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